Lecture 11. Determinants

Def Given a square matrix A, its <u>determinant</u> $\det(A)$ is defined by the following rules:

(i) If $A = [a]$ is an 1×1 matrix, $\det(A) = a$.

(ii) In general, if A is an $n\times n$ matrix we have

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + \cdots + (-1)^{n+1} a_{n1}\det(A_{n1})$$

where
- a_{ij} denotes the entry on row i and column j,
- A_{ij} denotes the matrix obtained by removing row i and column j.

Prop Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\det(A) = ad - bc$.

pf $a_{11} = a$, $a_{21} = c$, $A_{11} = [d]$, $A_{21} = [b]$

$\Rightarrow \det(A) = a\cdot \det[d] - c\cdot \det[b] = ad - bc$.

Note For an arbitrary square matrix, its determinant can be recursively computed by definition.

e.g. $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}$

$$\Rightarrow \det(A) = 3\det\underbrace{\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}}_{A_{11}} - 0\cdot\det\underbrace{\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}}_{A_{21}} + 1\cdot\det\underbrace{\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}}_{A_{31}}$$

$$= 3(2\cdot 5 - 1\cdot 4) - 0 + 1\cdot(0\cdot 1 - (-1)\cdot 2)$$

$$= 3\cdot 6 - 0 + 1\cdot 2 = 20$$

* Not recommended for matrices of large size.

Thm A square matrix A has an inverse $\iff \det(A) \neq 0$

Prop The determinant is multiplicative:

$$\det(AB) = \det(A)\det(B) \quad \text{for any } n\times n \text{ matrices } A, B$$

Def Let A be a square matrix.

(1) The main diagonal of A is the diagonal which runs from the top left corner to the bottom right corner.

(2) A is upper triangular if all entries below the main diagonal are zero.

(3) A is lower triangular if all entries above the main diagonal are zero.

(4) A is diagonal if all entries off the main diagonal are zero.

e.g. $\begin{bmatrix} 2 & 0 & -1 & 5 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 6 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 0 & 7 & 2 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

upper triangular — lower triangular — diagonal = upper & lower triangular

Prop If a square matrix A is triangular, $\det(A)$ is equal to the product of the diagonal entries.

Prop (Row operations and determinants)

(1) The addition operation does not change the determinant.

(2) The interchange operation changes the sign of the determinant.

(3) The multiplication operation multiplies the determinant by the multiplication factor.

Ex Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & 3 & 5 & 2 \\ 0 & 2 & 6 & -3 \\ 3 & 0 & -4 & 7 \end{bmatrix}$$

(1) Find det(A).

Sol We reduce A to a triangular matrix using row operations.

$$\det(A) \underset{\substack{R_2-2R_1 \\ R_4-3R_1}}{=} \det\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 3 & 9 & 0 \\ 0 & 2 & 6 & -3 \\ 0 & 0 & 2 & 4 \end{bmatrix} \underset{\frac{1}{3}R_2}{=} 3\det\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 6 & -3 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\underset{R_3-2R_2}{=} 3\det\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 4 \end{bmatrix} \underset{R_3 \leftrightarrow R_4}{=} -3\underbrace{\det\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}}_{\text{triangular}}$$

$$= -3 \cdot \underbrace{1 \cdot 1 \cdot 2 \cdot (-3)}_{\text{diagonal entries}} = \boxed{18}$$

Note We can similarly compute the determinant of an arbitrary square matrix using row operations.

(2) Determine whether A has an inverse

Sol $\det(A) \neq 0 \implies A$ $\boxed{\text{has an inverse}}$

Ex Consider the matrices

$$A = \begin{bmatrix} 3 & 3 & -2 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

(1) Find $\det(A)$ and $\det(B)$

Sol $$\det(A) = 3\det\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A_{11}} - 1\cdot\det\underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}}_{A_{21}} + 0\cdot\det\underbrace{\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}}_{A_{31}}$$

$$= 3(2\cdot2-1\cdot1) - 1\cdot(3\cdot2-(-2)\cdot1) + 0 = 1$$

$$\det(B) = 0\cdot\det\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{B_{11}} - 1\cdot\det\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{B_{21}} + 2\det\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{B_{31}}$$

$$= 0 - 1\cdot(1\cdot4-2\cdot3) + 2(1\cdot3-2\cdot2) = 0$$

Hence we have $\boxed{\det(A) = 1}$ and $\boxed{\det(B) = 0}$

(2) If possible, find a matrix X with $AX = B$

Sol $\det(A) \neq 0 \implies A$ has an inverse

$AX = B \implies \cancel{A^{-1}A}X = A^{-1}B \implies X = \underline{A^{-1}B}$ (order is important!)

$$\underbrace{\left[\begin{array}{ccc|ccc} 3 & 3 & -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]}_{A \quad\quad I} \xrightarrow{\text{RREF}} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 9 & -6 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 3 & -6 & 4 \end{array}\right]}_{\text{RREF}(A)=I \quad\quad A^{-1}}$$

$$\implies A^{-1} = \begin{bmatrix} -4 & 9 & -6 \\ -1 & 2 & -1 \\ 3 & -6 & 4 \end{bmatrix}$$

$$\implies X = \begin{bmatrix} -4 & 9 & -6 \\ -1 & 2 & -1 \\ 3 & -6 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 8 & 10 \\ -4 & -5 & -6 \\ 3 & 4 & 5 \end{bmatrix}}$$

(3) If possible, find a matrix Y with $YA=B$

Sol $YA=B \Rightarrow Y\cancel{AA^{-1}}=BA^{-1} \Rightarrow Y=BA^{-1}$ (order is important!)

$$\Rightarrow Y=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} -4 & 9 & -6 \\ -1 & 2 & -1 \\ 3 & -6 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 1 \\ 2 & -5 & 6 \\ 4 & -10 & 11 \end{bmatrix}}$$

(4) If possible, find a matrix Z with $BZ=A$

Sol $\det(BZ)=\underbrace{\det(B)}_{0}\det(Z)=0$ and $\det(A)=1$

$\Rightarrow \det(BZ)\neq\det(A) \Rightarrow BZ\neq A$

$\Rightarrow$ A matrix Z with $BZ=A$ $\boxed{\text{does not exist}}$

Note Our method in this example is comparable to how we solve a linear equation $ax=b$.

- $a\neq 0$: $a^{-1}=\frac{1}{a}$ exists $\Rightarrow \frac{1}{a}\cdot ax=\frac{1}{a}\cdot b \Rightarrow x=\frac{b}{a}$ (cf. (2), (3))
- $a=0,\ b\neq 0$: $0x=b\neq 0 \Rightarrow$ no solutions (cf. (4))
- $a=0,\ b=0$: $0x=0 \Rightarrow$ infinetely many solutions

* For matrix equations, the analogue of the last case is quite complicated. In fact, for $n\times n$ matrices A, B with determinant 0, the equation $AX=B$ may have no solutions or infinetely many solutions.